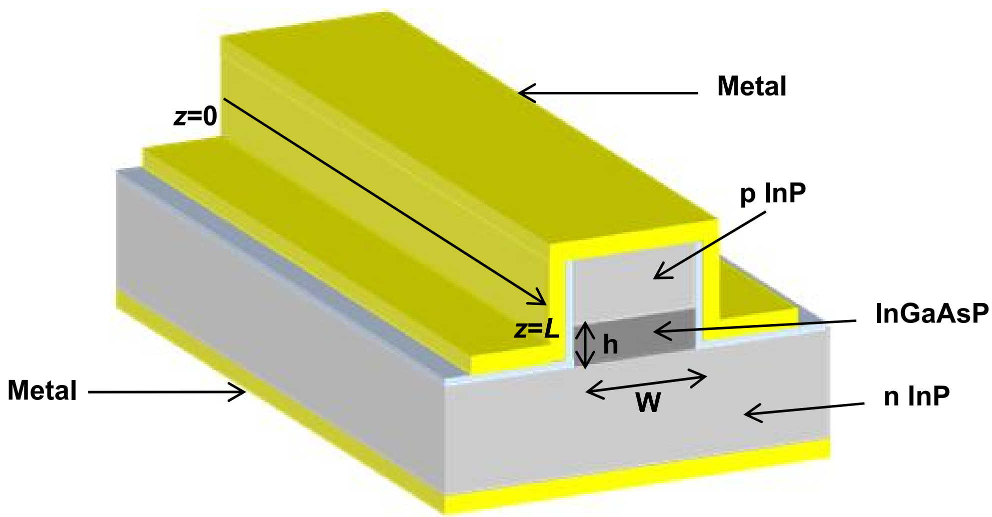
Chapter 9

Semiconductor Optical Amplifiers

9.1 Basic Structure of Semiconductor Optical Amplifiers (SOAs)

9.1.1 Introduction:
Semiconductor optical amplifiers (SOAs), as the name suggests, are used to amplify optical signals. A typical structure of a InGaAsP/InP SOA is shown in the Figure below. The basic structure consists of a heterostructure pin junction.



The smaller bandgap intrinsic region has smaller refractive index than the wider bandgap p-doped and n-doped quasineutral regions. The intrinsic region forms the core of the optical waveguide and the quasineutral regions form the claddings. Current injection into the intrinsic region (also called the active region) can create a large population of electrons and holes. If the carrier density exceeds the transparency carrier density then the material can have optical gain and the device can be used to amplify optical signals via stimulated emission. During operation as an optical amplifier, light is coupled into the waveguide at $z = 0$. As the light propagates inside the waveguide it gets amplified. Finally, when light comes out at $z = L$, its power is much higher compared to what it was at $z = 0$.

9.2 Basic Equations of Semiconductor Optical Amplifiers (SOAs)

9.2.1 Equation for the Optical Power:
The material gain of the active region can be described by a complex refractive index. Suppose the real part of the refractive index of the active region is n_a, the material group index of the active region n_{ag}^M, the group index of the waveguide optical mode is n_g, the material gain of the active region is g, and the mode confinement factor of the active region is Γ_a. Then the change in the propagation vector $\Delta\beta$ of the waveguide optical mode due to gain in the active region is given by the waveguide perturbation theory,

$$\Delta\beta = \frac{\omega}{c}\Gamma_a\left(\frac{n_g}{n_{ag}^M}\right)\Delta n_a = -i\,\Gamma_a\left(\frac{n_g}{n_{ag}^M}\right)\frac{g}{2} = -i\,\Gamma_a\,\frac{\tilde{g}}{2}$$

where,

$$\tilde{g} = \left(\frac{n_g}{n_{ag}^M}\right)g$$

In the presence of gain, the light field amplitude will increase with distance as $e^{\Gamma_a(\tilde{g}/2)z}$ and the optical power will increase as $e^{\Gamma_a\,\tilde{g}\,z}$. The factor $\Gamma_a\,\tilde{g}$ is called the modal gain. If $P(z)$ represents the optical power (units: energy per sec) then one can write a simple equation for the increase in the optical power with distance,

$$\frac{dP(z)}{dz} = \Gamma_a\,\tilde{g}\,P(z)$$

A time dependent form of the above equation for power propagating in the +z-direction will be,

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right)P(z,t) = \Gamma_a\,\tilde{g}\,P(z,t)$$

As the optical signal gets stronger with distance inside the waveguide, and the rate of stimulated emission also gets proportionally faster, the carrier density inside the active region also changes and cannot be assumed to be the same as in the absence of any optical signal inside the waveguide. In the next Section, we develop rate equations for the carrier density in the active region.

9.2.2 Modeling Waveguide Losses:

Material losses (such as those due to free carrier absorption) lead to losses in the waveguide mode. Suppose the material loss is represented by the function $\alpha(x,y)$. We can represent loss by the imaginary part of the refractive index. The change in the propagation vector due to loss is,

$$\Delta\beta = \omega\frac{\iint\left[\varepsilon_o n\,\Delta n\,\vec{E}.\vec{E}^*\right]dxdy}{\iint \mathrm{Re}\left[\vec{E}_t\times\vec{H}_t^*\right].\,\hat{z}\,dxdy} = \frac{ic}{2}\frac{\iint\left[\varepsilon_o n\,\alpha\,\vec{E}.\vec{E}^*\right]dxdy}{\iint \mathrm{Re}\left[\vec{E}_t\times\vec{H}_t^*\right].\,\hat{z}\,dxdy}$$

$$= \sum_k i\,\Gamma_k\left(\frac{n_g}{n_{kg}^M}\right)\frac{\alpha_k}{2} = \sum_k i\,\Gamma_k\,\frac{\tilde{\alpha}_k}{2} = \frac{\tilde{\alpha}}{2}$$

where the sum in the last line represents the sum over all the regions in the cross-section of the waveguide. The modal loss $\tilde{\alpha}$ is equal to the loss of each region weighted by its mode confinement factor. In the presence of loss, the equation for the optical power becomes,

$$\frac{dP(z)}{dz} = \left(\Gamma_a\,\tilde{g} - \tilde{\alpha}\right)P(z)$$

The time dependent form will be,

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right)P(z,t) = \left(\Gamma_a\,\tilde{g} - \tilde{\alpha}\right)P(z,t)$$

9.2.3 Rate Equation for the Carrier Density:

Recall from the discussion on LEDs that the rate equation for the carrier density in the active region of a pin heterostructure can be written as,

$$\frac{dn}{dt} = \frac{\eta_i\,I}{qV_a} - \left[R_{nr}(n) - G_{nr}(n)\right] - \left[R_r(n) - G_r(n)\right]$$

In the present case, the volume V_a of the active region is WhL and the cross-sectional area A_a of the active region is Wh. The radiative recombination-generation terms in the above equation include spontaneous emission into all (guided and unguided) radiation modes as well as stimulated emission and absorption by thermal photons in all (guided and unguided) radiation modes. Note that in the bandwidth of interest there will generally be many more unguided modes than guided modes. We assume that the density of radiation modes in the active region is not modified significantly from the expression valid for a bulk material and is given by,

$$g_p(\omega)=\left(\frac{\omega n_a}{\pi c}\right)^2 \frac{n_{ag}^M}{c}$$

The above approximation turns out to be fairly good even though the optical waveguide does modify the density of radiation modes from the expression given above.

We must now add stimulated emission and absorption from the guided optical mode to the right hand side of the above rate equation for the carrier density. Assuming the photon density in the active region is n_p, the net stimulated emission rate is,

$$R_{\downarrow}-R_{\uparrow}=\frac{c}{n_{ag}^M}g(n)n_p$$

The material gain $g(n)$ is carrier density dependent and may be approximated as,

$$g(n)=g_o \ln\left(\frac{n}{n_{tr}}\right)$$

The values of the transparency carrier density n_{tr} range from 1.5x10^{18} 1/cm^3 to 3.0x10^{18} 1/cm^3 and the values of g_o range from 1000 to 4000 /1cm for most III-V materials. The carrier density rate equation becomes,

$$\frac{dn}{dt}=\frac{\eta_i I}{qV_a}-[R_{nr}(n)-G_{nr}(n)]-[R_r(n)-G_r(n)]-\frac{c}{n_{ag}^M}g(n)n_p$$

It is better to write the last term on the right hand side in terms of $\tilde{g}$ where,

$$\tilde{g}=\left(\frac{n_g}{n_{ag}^M}\right)g$$

and we get,

$$\frac{dn}{dt}=\frac{\eta_i I}{qV_a}-[R_{nr}(n)-G_{nr}(n)]-[R_r(n)-G_r(n)]-v_g\,\tilde{g}(n)n_p$$

Note that now the group velocity of the optical mode appears in the last term on the right hand side. In the above equation, both the carrier density and the photon density are functions of position inside the waveguide. More explicitly,

$$\frac{dn(z,t)}{dt}=\frac{\eta_i I}{qV_a}-[R_{nr}(n(z,t))-G_{nr}(n(z,t))]-[R_r(n(z,t))-G_r(n(z,t))]-v_g\,\tilde{g}(n(z,t))n_p(z,t)$$

We need to relate the photon density n_p inside the active region to the optical power P. Since the mode confinement factor Γ_a is the ratio of the average mode energy density (units: energy per unit length) inside the active region to the average mode energy density W (units: energy per unit length) in the entire waveguide,

$$n_p A_a=\Gamma_a \frac{W}{\hbar\omega}$$

But, $P=v_g W$, therefore,

$$n_p A_a=\Gamma_a \frac{P}{\hbar\omega v_g}$$

The **effective area A_{eff} of the optical mode** is defined by the relation,

$$A_{eff} = \frac{A_a}{\Gamma_a}$$

The above definition implies that the photon density in the active region can also be written as,

$$n_p = \frac{P}{\hbar\omega v_g A_{eff}}$$

We can now write the carrier density rate equation as,

$$\frac{dn(z,t)}{dt} = \frac{\eta_i I}{qV_a} - \left[R_{nr}(n(z,t)) - G_{nr}(n(z,t))\right] - \left[R_r(n(z,t)) - G_r(n(z,t))\right] - \tilde{g}(n(z,t))\frac{P(z,t)}{\hbar\omega A_{eff}}$$

The above equation together with,

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right)P(z,t) = \left[\Gamma_a \tilde{g}(n(z,t)) - \tilde{\alpha}\right]P(z,t)$$

are the two basic equations used to analyze semiconductor optical amplifiers.

9.3 Operation of Semiconductor Optical Amplifiers (SOAs)

9.3.1 Case I – No Gain Saturation:

We assume that the SOA is operating in steady state with an extremely small light signal input to the SOA at $z = 0$. We assume that $P(z = 0)$ is so small that $P(z)$ for all z, even after amplification, remains small and, consequently, $n_p(z)$ is also small. By small I mean small enough such that one may ignore the stimulated emission term in the carrier density rate equation compared to the other recombination-generation terms. In this case, the steady state carrier density is independent of position and can be obtained from the equation,

$$0 = \frac{\eta_i I}{qV_a} - \left[R_{nr}(n) - G_{nr}(n)\right] - \left[R_r(n) - G_r(n)\right]$$

Once the carrier density is determined, the material gain can be obtained using,

$$g(n) = g_o \ln\left(\frac{n}{n_{tr}}\right)$$

In steady state, the equation for the optical power becomes,

$$\frac{\partial P(z)}{\partial z} = \left[\Gamma_a \tilde{g}(n) - \tilde{\alpha}\right]P(z,t)$$

$$\Rightarrow P(z) = P(0)e^{\left[\Gamma_a \tilde{g}(n) - \tilde{\alpha}\right]z}$$

The dimensionless gain G of the amplifier is defined as the ratio of the output power to the input power,

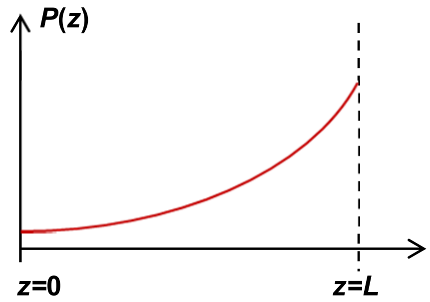
$$G = \frac{P(L)}{P(0)} = e^{\left[\Gamma_a \tilde{g} - \tilde{\alpha}\right]L}$$

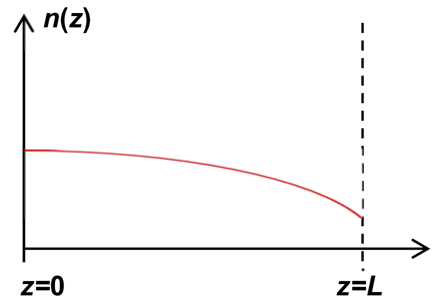
The amplifiers gain is usually specified in dB scale,

Gain in dB = $10\log_{10}(G)$

9.3.2 Case II – Gain Saturation:

In the more general case, stimulated emission term in the carrier density rate equation cannot be ignored. If either the input optical power is large or if the modal gain $\Gamma_a \tilde{g}$ is large, the photon density $n_p(z)$ can also be very large, especially near the output end of the amplifier ($z = L$). A large photon density increases the rate of carrier recombination by stimulated emission. Since photon density $n_p(z)$ is z-dependent, the carrier density $n(z)$ in steady state will also be z-dependent. The situation will look as follows,





The carrier density, and consequently the gain $\tilde{g}$, are both reduced near $z = L$. This is called "gain saturation"; light which is amplified by a gain medium ends up reducing the gain of that medium. In other words light starts "eating" the hand that feeds it. Gain saturation makes the amplifier nonlinear.

9.3.3 Input-Output Characteristics of SOAs – A Simple Solvable Model:

The complete non-linear equations of an SOA are difficult to solve analytically. However, with certain approximations, an analytic solution can be obtained. We assume that the material gain can be approximated by a linear model,

$$\tilde{g}(n) = \tilde{g}_o \ln\left(\frac{n}{n_{tr}}\right) \approx \left.\frac{d\tilde{g}_o}{dn}\right|_{n=n_{tr}} (n - n_{tr}) = \tilde{a}_o\,(n - n_{tr})$$

The linear model holds well at least for carrier densities near the transparency carrier density. The quantity $\tilde{a}_o$ is called the differential gain (units: cm^2). We also assume that the recombination-generation rates can also be approximated with a linear model,

$$[R_{nr}(n) - G_{nr}(n)] + [R_r(n) - G_r(n)] \approx \frac{n - n_i}{\tau_r} \approx \frac{n}{\tau_r}$$

Here, τ_r is a phenomenological recombination time. With these approximations we can write the following set of equations for operation in the steady state,

$$\frac{dP(z)}{dz} = \left[\Gamma_a\,\tilde{a}_o\,(n(z) - n_{tr}) - \tilde{\alpha}\,\right] P(z) \qquad (1)$$

$$\frac{\eta_i\,I}{qV_a} = \frac{n(z)}{\tau_r} + \tilde{a}_o\,(n(z) - n_{tr})\,\frac{P(z)}{\hbar\omega\,A_{eff}}$$

The second equation gives us,

$$n(z) - n_{tr} = \frac{\dfrac{\eta_i\,I\tau_r}{qV_a} - n_{tr}}{1 + \tilde{a}_o\tau_r\,\dfrac{P(z)}{\hbar\omega\,A_{eff}}}$$

The above equation shows that the reduction of the carrier density and the saturation of the gain is governed by the denominator. We write the above expression as,

$$n(z) - n_{tr} = \frac{\dfrac{\eta_i I \tau_r}{q V_a} - n_{tr}}{1 + \dfrac{P(z)}{P_{sat}}} \qquad (3)$$

where,

$$P_{sat} = \frac{\hbar\omega A_{eff}}{\tilde{a}_o \tau_r}$$

The quantity P_{sat} defines the optical power at which gain saturation cannot be ignored. When $P(z) << P_{sat}$ gain saturation can be ignored and carrier density can be determined assuming the optical power is zero. The unsaturated value of the modal gain is,

$$\Gamma_a \tilde{g}^* = \Gamma_a \, \tilde{a}_o (n(z) - n_{tr}) \, |_{P(z)<<P_{sat}} = \Gamma_a \, \tilde{a}_o \left(\frac{\eta_i I \tau_r}{q V_a} - n_{tr} \right)$$

and the unsaturated value of the amplifier gain is,

$$G^* = e^{\left(\Gamma_a \tilde{g}^* - \tilde{\alpha}\right)L}$$

Plugging the result in (3) into (1) gives,

$$\frac{dP(z)}{dz} = \left[\frac{\Gamma_a \tilde{g}^*}{1 + \dfrac{P(z)}{P_{sat}}} - \tilde{\alpha} \right] P(z)$$

It is clear from the above equation that if $P(z) << P_{sat}$ then the amplifier gain is just the unsaturated gain $G^* = e^{\left(\Gamma_a \tilde{g}^* - \tilde{\alpha}\right)L}$. Solution of the above equation via direct integration gives,

$$\frac{P(0)}{P_{sat}} = \left(\frac{\Gamma_a \tilde{g}^*}{\tilde{\alpha}} - 1 \right) \left(\frac{e^{\frac{\left(\Gamma_a \tilde{g}^* - \tilde{\alpha}\right)L - \ln G}{\Gamma_a \tilde{g}^* / \tilde{\alpha}}} - 1}{G \, e^{\frac{\left(\Gamma_a \tilde{g}^* - \tilde{\alpha}\right)L - \ln G}{\Gamma_a \tilde{g}^* / \tilde{\alpha}}} - 1} \right) = \left(\frac{\Gamma_a \tilde{g}^*}{\tilde{\alpha}} - 1 \right) \left(\frac{\left(\dfrac{G^*}{G} \right)^{\tilde{\alpha}/\Gamma_a \tilde{g}^*} - 1}{G \left(\dfrac{G^*}{G} \right)^{\tilde{\alpha}/\Gamma_a \tilde{g}^*} - 1} \right)$$

In the above equation G is the amplifier gain defined as $P(L)/P(0)$. The above equation can be used to obtain G as a function of the unsaturated modal gain and the input optical power. Since the amplifier gain depends on the input power, the amplifier is nonlinear. The nonlinearity is due to gain saturation. When $P(0) << P_{sat}$ the amplifier gain G equals the unsaturated value G^*. As the input power $P(0)$ increases, the optical power $P(L)$ at the output becomes large enough to cause a significant reduction in the carrier density $n(z)$ close to $z = L$, and when the carrier density decreases, the gain G, which can also be written as,

$$G = e^{\int_0^L \left(\Gamma_a \tilde{a}_o \Delta n(z) - \tilde{\alpha}\right) dz}$$

also decreases. This is gain saturation. Two important figures of merit of SOAs are the input saturation power and the output saturation power. The input saturation power is the input optical power at which the amplifier gain G decreases by a factor of two (or by 3 dB) from the unsaturated value G^*. The output saturation power is the output optical power at which the amplifier gain decreases by a factor of two (or by 3 dB). The input saturation power is given by the expression,

$$\frac{P(0)}{P_{sat}} = \left(\frac{\Gamma_a \tilde{g}^*}{\tilde{\alpha}} - 1\right)\left(\frac{2^{\tilde{\alpha}/\Gamma_a \tilde{g}^*} - 1}{(G^*/2)\,2^{\tilde{\alpha}/\Gamma_a \tilde{g}^*} - 1}\right)$$

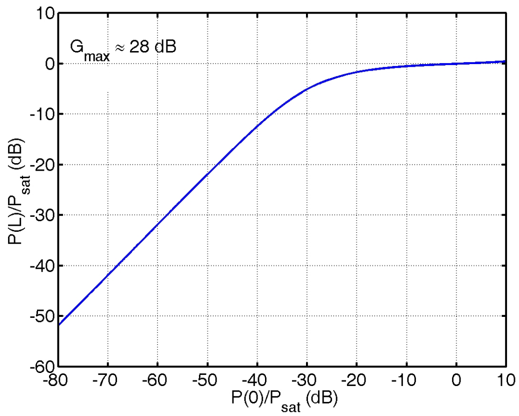
The output saturation power is,

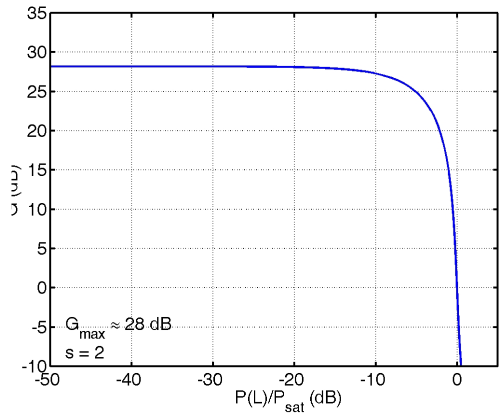
$$\frac{P(L)}{P_{sat}} = \left(\frac{\Gamma_a \tilde{g}^*}{\tilde{\alpha}} - 1\right)\left(\frac{G^*}{2}\right)\left(\frac{2^{\tilde{\alpha}/\Gamma_a \tilde{g}^*} - 1}{(G^*/2)\,2^{\tilde{\alpha}/\Gamma_a \tilde{g}^*} - 1}\right)$$

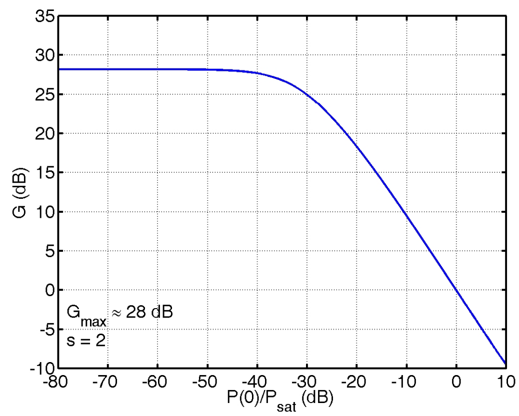
The maximum output saturation power the amplifier can produce is obtained by taking the limit $G^* \rightarrow \infty$ assuming that the ratio $\Gamma_a \tilde{g}^*/\tilde{\alpha}$ remains constant. For example, G^* can be increased by increasing the length of the amplifier. The maximum output saturation power is,

$$\frac{P(L)}{P_{sat}} = \left(\frac{\Gamma_a \tilde{g}^*}{\tilde{\alpha}} - 1\right)\left(1 - 2^{-\tilde{\alpha}/\Gamma_a \tilde{g}^*}\right)$$

The above equation shows that the maximum value of the output saturation power is of the order of P_{sat}. More insight can be obtained by plotting $P(L)$ vs $P(0)$ and the gain G vs the output power $P(L)$ and vs the input power $P(0)$. These graphs are shown below for G^* equal to 28 dB and the ratio $\Gamma_a \tilde{g}^*/\tilde{\alpha}$ equal to 2. All the quantities are plotted in decibels (dB).







The plots show:

i) the decrease in the amplifier gain with the input optical power when the input optical power exceeds P_{sat}/G^*.

ii) the saturation of the output optical power at large input powers to values close to P_{sat}.

SOAs with large output saturation powers are desirable. In order to increase the output saturation power one must increase the value of P_{sat} and the value of the ratio $\Gamma_a \tilde{g}^* / \tilde{\alpha}$.

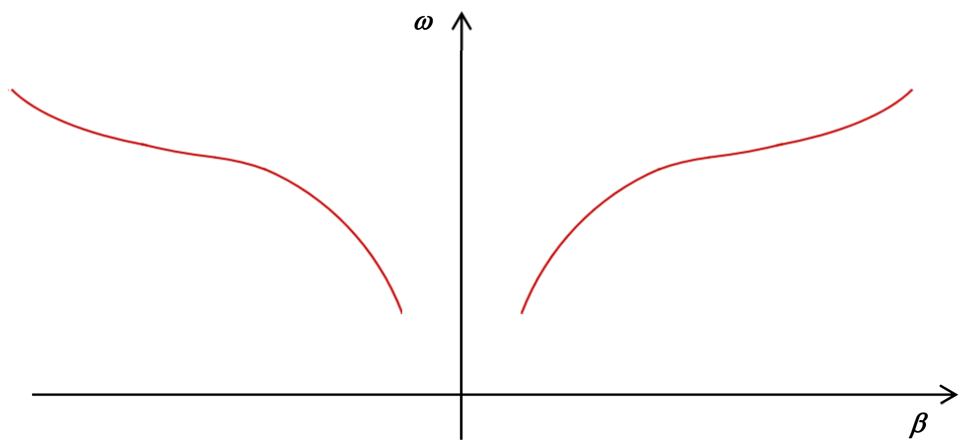
9.4 Amplified Spontaneous Emission (ASE) in Semiconductor Optical Amplifiers (SOAs)

9.4.1 Introduction:

Spontaneously emitted photons into all the unguided radiation modes leave the active region soon after emission. Spontaneously emitted photons into the guided radiation mode travel along the waveguide and get amplified via stimulated emission. This amplified spontaneous emission (ASE) exits from the output end of the amplifier along with the amplified input signal. ASE is undesirable but unavoidable. It is considered a part of the noise added by the optical amplifier.

9.4.2 Amplified Spontaneous Emission:

Suppose the optical waveguide of the SOA supports only a single guided mode. When we say a "single mode waveguide" we do not mean that only a single radiation mode is guided. What we mean is that the waveguide only supports a single transverse optical mode. For this single transverse mode, the propagation vector $\beta(\omega)$ is a function of frequency, as shown below, and different values of $\beta(\omega)$ correspond to different longitudinal modes of the waveguide.



If the length of the waveguide is L then periodic boundary conditions give $(L/2\pi)\Delta\beta$ different longitudinal modes in an interval $\Delta\beta$. From previous Chapters we know how to calculate the spontaneous emission rate into a single radiation mode. The expression for the spontaneous emission rate has the same form as that for the stimulated emission rate except that the photon occupation of the mode is taken to be unity. The spontaneous emission rate into a longitudinal mode of frequently ω per unit volume of the active region per second is,

$$v_g \, \tilde{g}(\omega) n_{sp}(\omega) \frac{1}{V_p}$$

Here, V_p is the modal volume of the mode and equals $A_{eff} L$. To proceed further, we will make some assumptions that will simplify things. We assume that:

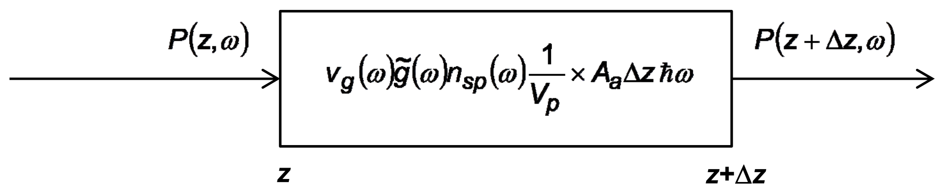
a) There is no input optical signal.
b) The photons travelling in the waveguide are entirely due to spontaneous emission and amplified spontaneous emission and not coming from any input signal or amplified input signal.

c) The photon density everywhere in the waveguide is small enough to not cause any significant reduction in the carrier density due to stimulated recombination. Consequently, carrier density can be calculated as if there were no photons in the waveguide.

Knowing the carrier density, we can calculate the gain $\tilde{g}(\omega)$ and the spontaneous emission factor $n_{sp}(\omega)$ which are both functions of the photon frequency ω. As assumed, the gain will be the unsaturated gain $\tilde{g}^*(\omega)$.

Suppose the ASE optical power at frequency ω moving in the +z-direction is given by $P(z,\omega)$. Consider a small waveguide segment of length Δz located at z. The increase in power from z to $z+\Delta z$ due to the addition of spontaneously emitted photons is,

$$P(z+\Delta z,\omega)-P(z,\omega)=v_g(\omega)\tilde{g}(\omega)n_{sp}(\omega)\frac{1}{V_p}\times A_a\Delta z\,\hbar\omega$$



This implies,

$$\frac{\partial P(z,\omega)}{\partial z}=\Gamma_a\,\tilde{g}(\omega)n_{sp}(\omega)\hbar\omega\left(\frac{v_g}{L}\right)$$

The above equation contains only the spontaneous emission contribution. We also add the stimulated emission-absorption and loss contributions to get,

$$\frac{\partial P(z,\omega)}{\partial z}=\left(\Gamma_a\tilde{g}(\omega)-\tilde{\alpha}\right)P(z,\omega)+\Gamma_a\,\tilde{g}(\omega)n_{sp}(\omega)\hbar\omega\left(\frac{v_g}{L}\right)$$

The solution subject to the boundary condition $P(z=0,\omega)=0$ is,

$$P(z=L,\omega)=\hbar\omega\left(\frac{v_g}{L}\right)\frac{\Gamma_a\tilde{g}(\omega)}{\Gamma_a\tilde{g}(\omega)-\tilde{\alpha}}n_{sp}(\omega)\left(e^{\left(\Gamma_a\tilde{g}(\omega)-\tilde{\alpha}\right)L}-1\right)$$

Note that the ASE power is roughly proportional to the gain $G(\omega)=e^{\left(\Gamma_a\tilde{g}(\omega)-\tilde{\alpha}\right)L}$ of the amplifier.

The above expression gives the ASE power at the output ($z=L$) in only one longitudinal radiation mode. To get the total ASE power coming out at $z=L$ we need to sum the power in all the longitudinal modes,

$$P_{ASE}=L\int_0^\infty\frac{d\beta}{2\pi}P(z=L,\omega)=L\int_0^\infty\frac{d\beta}{2\pi}\hbar\omega\left(\frac{v_g}{L}\right)\frac{\Gamma_a\tilde{g}(\omega)}{\Gamma_a\tilde{g}(\omega)-\tilde{\alpha}}n_{sp}(\omega)\left(e^{\left(\Gamma_a\tilde{g}(\omega)-\tilde{\alpha}\right)L}-1\right)$$

We can convert the above integral into a frequency integral by noting that,

$$\frac{1}{v_g}=\frac{\partial\beta}{\partial\omega}$$

and get,

$$P_{ASE}=L\int_0^\infty\frac{d\omega}{2\pi}\frac{1}{v_g}P(z=L,\omega)=\int_0^\infty\frac{d\omega}{2\pi}\hbar\omega\,\frac{\Gamma_a\tilde{g}(\omega)}{\Gamma_a\tilde{g}(\omega)-\tilde{\alpha}}n_{sp}(\omega)\left(e^{\left(\Gamma_a\tilde{g}(\omega)-\tilde{\alpha}\right)L}-1\right)$$

The integral is non-zero and significant only within a bandwidth roughly equal to the gain bandwidth. For frequencies at which $\tilde{g}(\omega)=0$, $n_{sp}(\omega)$ is infinite, but the product $\tilde{g}(\omega)n_{sp}(\omega)$ is always finite, and therefore the integrand is also finite. An equal amount of ASE power comes out from the input end of the amplifier.

Usually an optical filter is placed in front of the SOA to cut down the ASE in unused bandwidth. Suppose the filter has a center frequency ω_f and a band width $\Delta\omega_f$. Then the ASE power going through the filter is,

$$P_{ASE} = \left(\frac{\Delta\omega_f}{2\pi}\right)\hbar\omega_f \; \frac{\Gamma_a \tilde{g}(\omega_f)}{\Gamma_a \tilde{g}(\omega_f) - \tilde{\alpha}} n_{sp}(\omega_f)\left(e^{\left(\Gamma_a \tilde{g}(\omega_f) - \tilde{\alpha}\right)L} - 1\right)$$